Topological excitations in 2D spin system with high spin $s \geqslant 1$

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Abstract

We construct a class of topological excitations of a mean field in a two-dimensional spin system represented by a quantum Heisenberg model with high powers of exchange interaction. The quantum model is associated with a classical one (the continuous classical analogue) that is based on a Landau-Lifshitz like equation, and describes large-scale fluctuations of the mean field. On the other hand, the classical model is a Hamiltonian system on a coadjoint orbit of the unitary group $\mathrm{SU}(2s+1)$ in the case of spin s. We have found a class of mean field configurations that can be interpreted as topological excitations, because they have fixed topological charges. Such excitations change their shapes and grow preserving an energy.

1 Introduction

According to Mermin and Wagner [1] there is no ferromagnetic or antiferromagnetic order in the one- and two-dimensional isotropic Heisenberg models with interactions of finite range at nonzero temperature. This statement is proven due to Bogoliubov's inequality in the general case. Here we construct excitations that cause a destruction of a long-range nematic or mixed ferromagnetic-nematic order. This is an extension of the results of Belavin and Polyakov [2].

We model a planar magnet by a square atomic lattice with the same spin s at each site. We describe this two-dimensional spin system by a generalized Heisenberg Hamiltonian, taking into account high powers of the exchange interaction $(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_{n+\delta})$, where $\hat{\boldsymbol{S}}_n$ is a vector of spin operators at site n. By a mean field approximation we obtain a classical long-range equation from the quantum Heisenberg one.

An equation for a mean field (the field of magnetization and multipole moments) is a Hamiltonian equation on a coadjoint orbit of Lie group. At the same time, this is a generalization of the well-known Landau-Lifshitz equation for a magnetization field. In this context we obtain effective Hamiltonians for the magnetic system in question. Using Kählerian structure of coadjoint orbits, we construct effective Hamiltonians such that their minimums are proportional to topological charges of excitations. In addition, we produce these mean field configurations that give minimums to the Hamiltonians.

2 Quantum and classical models

As mentioned above, we represent the spin system by a planar atomic lattice with the same spin s at all sites. We assign three spin operators $(\hat{S}_n^1, \hat{S}_n^2, \hat{S}_n^3) = \hat{\boldsymbol{S}}_n$ to each atom n; the operators obey the standard commutation relations

$$[\hat{S}_n^a, \hat{S}_m^b] = i\varepsilon_{abc}\hat{S}_n^c\delta_{nm},$$

where a, b, c run over the values $\{1, 2, 3\}$, and δ_{nm} is the Kronecker symbol.

2.1 Generalized Heisenberg Hamiltonians

We are interested in so called high spins $s \geqslant 1$. In this case, we can describe the system by the following *bilinear-biquadratic Hamiltonian*

$$\hat{\mathcal{H}}^2 = -\sum_{n,\delta}\Big(J(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_{n+\delta}) + K(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_{n+\delta})^2\Big).$$

Here δ runs over the nearest-neighbour sites, n runs over all sites of the lattice, constants J and K denote exchange integrals. In the case of spin $s \geqslant 3/2$, the above Hamiltonian can include also the bicubic exchange, namely

$$\hat{\mathcal{H}}^3 = -\sum_{n,\delta}\Big(J(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_{n+\delta}) + K(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_{n+\delta})^2 + L(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_{n+\delta})^3\Big),$$

where L denotes the corresponding exchange integral.

One can easily write a generalized Heisenberg Hamiltonian for the system of an arbitrary spin s or greater than s. This Hamiltonian contains all powers of the exchange interaction up to $2s$. It can be reduced to a bilinear form if one takes the $2s+1$-dimensional space of irreducible representation of the group SU(2). The spin operators $\{\hat{S}_n^a\}$ over this space generate a complete associative matrix algebra, which has a sufficient number of operators to reduce the corresponding Hamiltonian to a bilinear form.

For example, in the case of spin $s = 1$ the appropriate space of representation is 3-dimensional, and we choose a canonical basis in the form: $\{|{+1}\rangle, |{-1}\rangle, |0\rangle\}$. The spin operators $\{\hat{S}_n^a\}$ generate the algebra $\mathrm{Mat}_{3\times 3}$. In order to form a basis in the algebra we take the tensor operators of weight 2

$$\begin{aligned} &\hat{Q}_n^{ab} = \hat{S}_n^a\hat{S}_n^b + \hat{S}_n^b\hat{S}_n^a, \quad a \neq b, \\ &\hat{Q}_n^{22} = (\hat{S}_n^1)^2 - (\hat{S}_n^2)^2, \qquad \hat{Q}_n^{20} = \sqrt{3}\big((\hat{S}_n^3)^2 - \tfrac{2}{3}\big) \end{aligned} \tag{1}$$

in addition to the spin operators. The introduced operators are called *quadrupole operators*.

In the case of spin $s = 3/2$ the appropriate space of representation is 4-dimensional, and $\{|{+\tfrac{3}{2}}\rangle, |{+\tfrac{1}{2}}\rangle, |{-\tfrac{1}{2}}\rangle, |{-\tfrac{3}{2}}\rangle\}$ is a canonical basis. We complete the associative matrix algebra $\mathrm{Mat}_{4\times 4}$ of $\{\hat{S}_n^a\}$ by the tensor operators of weights 2

and 3, defining them by the following formulas:

$$\begin{aligned}
\hat{Q}^{ab}_n &= \tfrac{\sqrt{5}}{2\sqrt{3}}\Big(\hat{S}^a_n\hat{S}^b_n + \hat{S}^b_n\hat{S}^a_n\Big), \quad a \neq b \\
\hat{Q}^{22}_n &= \tfrac{\sqrt{5}}{2\sqrt{3}}\Big((\hat{S}^1_n)^2 - (\hat{S}^2_n)^2\Big), \quad \hat{Q}^{20}_n = \tfrac{\sqrt{5}}{2}\big((\hat{S}^3_n)^2 - \tfrac{5}{4}\big),
\end{aligned} \tag{2}$$

$$\begin{aligned}
\hat{T}^{a3}_n &= (\hat{Q}^{a2}_n\hat{S}^3_n + \hat{S}^3_n\hat{Q}^{a2}_n), \quad a,b \in \{1,2\},\ a \neq b, \\
\hat{T}^{ab}_n &= \tfrac{1}{\sqrt{6}}\Big((\hat{S}^a_n)^2\hat{S}^b_n + \hat{S}^b_n(\hat{S}^a_n)^2 + \hat{S}^a_n\hat{S}^b_n\hat{S}^a_n - (\hat{S}^b_n)^3\Big), \\
\hat{T}^{3a}_n &= \tfrac{1}{\sqrt{10}}\Big(\hat{Q}^{a3}_n\hat{S}^3_n + \hat{S}^3_n\hat{Q}^{a3}_n + \sqrt{3}(\hat{Q}^{20}_n\hat{S}^a_n + \hat{S}^a_n\hat{Q}^{20}_n)\Big), \\
\hat{T}^{30}_n &= \tfrac{1}{12}\big(41\hat{S}^3_n - 20(\hat{S}^3_n)^3\big).
\end{aligned} \tag{3}$$

We call the tensor operators of weight 3 *sextupole operators*. In what follows we denote all tensor operators over the chosen space of representation by $\{\hat{P}^a_n\}$.

In terms of representation operators a generalized Heisenberg Hamiltonian gets a bilinear form. For the Hamiltonians considered above we have:

$$\begin{aligned}
\hat{\mathcal{H}}^2 &= -(J - \tfrac{1}{2}K)\sum_{n,\delta}\sum_{b}\hat{S}^b_n\hat{S}^b_{n+\delta} - \tfrac{1}{2}K\sum_{n,\delta}\sum_{\alpha}\hat{Q}^\alpha_n\hat{Q}^\alpha_{n+\delta} - \tfrac{4}{3}KN; \\
\hat{\mathcal{H}}^3 &= -(J - \tfrac{1}{2}K + \tfrac{587}{80}L)\sum_{n,\delta}\sum_{b}\hat{S}^b_n\hat{S}^b_{n+\delta} - \tfrac{75}{32}(4K - L)N - \\
&\qquad - \tfrac{6}{5}(K - 2L)\sum_{n,\delta}\sum_{\alpha}\hat{Q}^\alpha_n\hat{Q}^\alpha_{n+\delta} - \tfrac{9}{10}L\sum_{n,\delta}\sum_{\beta}\hat{T}^\beta_n\hat{T}^\beta_{n+\delta},
\end{aligned}$$

where N is the overall number of sites in the lattice. Obviously, the obtained bilinear Hamiltonians are SU(2)-invariant, because they are constructed from representation operators of the group SU(2).

2.2 Mean field approximation

Here a mean field is a field of expectation values for the operators $\{\hat{P}^a_n\}$ calculated after spontaneous breaking of symmetry. The breaking of symmetry is performed by switching on an external magnetic field, which specifies an order in the system; then the external field vanishes. Such kind of averages is also called *quasiaverages* [3].

We denote a mean field averaging by $\langle\cdot\rangle$, and components of the mean field by $\{\mu_a(\boldsymbol{x}_n) = \langle\hat{P}^a_n\rangle\}$. A mean field approximation of the bilinear-biquadratic Hamiltonian has the form

$$\hat{\mathcal{H}}^2_{\mathrm{MF}} = -(J - \tfrac{1}{2}K)z\sum_n\sum_{a=1}^{3}\hat{P}^a_n\mu_a(\boldsymbol{x}_n) - \tfrac{1}{2}Kz\sum_n\sum_{a=4}^{8}\hat{P}^a_n\mu_a(\boldsymbol{x}_n) - \tfrac{4}{3}KzN,$$

where z is a number of the nearest-neighbour sites.

Evidently, a mean field Hamiltonian remains SU(2)-invariant. Then by an action of the group SU(2) it can be reduced to a diagonal form. Of course,

this reduction is possible only in the case of thermodynamical equilibrium and an infinite lattice, when the mean field becomes constant and the dependance on site n can be omitted. Moreover, almost all components of the mean field vanish, except the components corresponding to diagonal operators of $\{\hat{P}_n^a\}$. For the bilinear-biquadratic Hamiltonian a diagonal form is the following:

$$\hat{\mathcal{H}}^2_{\mathrm{MF}} = -zN\left((J - \tfrac{1}{2}K)\hat{S}^3\mu_3 + \tfrac{1}{2}K\hat{Q}^{20}\mu_8 + \tfrac{4}{3}K\right).$$

The remaining components are suitable to serve as *order parameters*. The component μ_3 describes a normalized *magnetization* (a ratio of the z-projection of magnetic moment to a saturation magnetization). The components μ_8 and μ_{15} are normalized *projections of quadrupole* and *sextupole moments*, respectively.

Possible values of order parameters can be obtained from the *self-consistent equations*

$$\mu_a = \langle \hat{P}^a \rangle_{\mathrm{MF}} = \frac{\mathrm{Tr}\,\hat{P}^a e^{-\frac{\hat{h}_{\mathrm{MF}}}{kT}}}{\mathrm{Tr}\,e^{-\frac{\hat{h}_{\mathrm{MF}}}{kT}}}$$

for all diagonal operators $\hat{P}^a$. Here we use the density matrix with the one-site Hamiltonian $\hat{h}_{\mathrm{MF}} = \hat{\mathcal{H}}_{\mathrm{MF}}/N$. Note, that for the standard Heisenberg Hamiltonian, when only the spin operators are considered, a self-consistent equation turns into the well-known Weiss equation.

In the case of bilinear-biquadratic Hamiltonian, an analysis of self-consistent equations gives the following. We adduce probable values of order parameters in the limit $T \to 0$ as $J,\, K > 0$: 1) $|\mu_3| = 1,\ \mu_8 = \frac{2J-K}{\sqrt{3}\,K}$; 2) $|\mu_3| = \frac{1}{2},\ \mu_8 = \frac{J-K/2}{\sqrt{3}\,K}$; 3) $\mu_3 = 0,\ |\mu_8| = \frac{2}{\sqrt{3}}$; 4) $\mu_3 = 0,\ |\mu_8| = \frac{1}{\sqrt{3}}$. Two of the solutions correspond to states with the ferromagnetic order, because they have a nonzero magnetization μ_3. The other two solutions correspond to states with the quadrupole order (so called nematic states), which are states with zero magnetization. Solutions 2) and 4) are unstable and called partly ordered. If K becomes negative, only solution 1) remains. These results accord with the results of [4, 5] and with the phase diagram of ordered states in a one-dimensional spin system from [6].

In what follows we consider an SU(3)-invariant system with the bilinear-biquadratic Hamiltonian. The SU(3)-invariance is reached by assigning $J = K$; this is the boundary line between the ferromagnetic and the nematic regions (see [6]). It means the system can appear in the both states. In the case of Hamiltonian $\hat{\mathcal{H}}^3$, the maximal SU(4)-invariance is reached as $J = -\frac{81}{44}\,K = -\frac{81}{16}\,L$, that is located within the ferromagnetic region.

Now we apply the mean field averaging to the quantum Heisenberg equation

$$i\hbar \frac{d\hat{P}_n^a}{dt} = [\hat{P}_n^a, \hat{\mathcal{H}}]. \tag{4}$$

The averaging is performed with the assumption of zero correlations between fluctuations of $\{\hat{P}_n^a\}$ at distinct sites: $\langle \hat{P}_n^a \hat{P}_m^b \rangle = \langle \hat{P}_n^a \rangle \langle \hat{P}_m^b \rangle$. Then we take a large-scale limit and obtain the Landau-Lifshitz like equation

$$\hbar \frac{\partial \mu_a}{\partial t} = 2Jl^2\, C_{abc}\mu_b(\mu_{c,xx} + \mu_{c,yy}), \tag{5}$$

where C_{abc} are structure constants of the Lie algebra of $\{\hat{P}_n^a\}$ with the commutation relations $[\hat{P}_n^a, \hat{P}_m^b] = iC_{abc}\hat{P}_n^c\delta_{nm}$. Equation (5) is an equation of motion for the mean field $\{\mu_a(\boldsymbol{x})\}$ over the plain $\{\boldsymbol{x} = (x, y) \mid x, y \in \mathbb{R}\}$ that replaces the lattice.

In the case of standard Heisenberg Hamiltonian (5) coincides with the Landau-Lifshitz equation for an isotropic magnet. Therefore, in the general case we call (5) a *generalized Landau-Lifshitz equation* for the vector field $\{\mu_a\}$. The vector field has 8 components if one exploits the bilinear-biquadratic Hamiltonian, and 15 components for the Hamiltonian with the bicubic exchange.

2.3 Effective Hamiltonians on coadjoint orbits

The generalized Landau-Lifshitz equation (5) can be interpreted as a *Hamiltonian equation on a coadjoint orbit of Lie group.* In the case of spin $s=1$ we deal with the group SU(3), in the case of an arbitrary spin s the group is SU($2s$+1). Note, that the matrices $\{\hat{P}^a\}$ serve as a basis in the corresponding Lie algebra $\mathfrak{su}(2s+1)$, and components of the mean field $\{\mu_a\}$ serve as coordinates in the dual space to $\mathfrak{su}(2s+1)$.

We start with brief description of the groups SU(3) and SU(4). For more material see, in particular, [7, 8].

The group SU(3) has two types of orbits: the generic $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$ of dimension 6, and the degenerate $\frac{\mathrm{SU}(3)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$ of dimension 4. Each orbit of SU(3) is defined by two numbers m and q, which are values of the coordinates μ_3 and μ_8 at an initial point (a point in the positive Weyl chamber). Simultaneously, these numbers are limiting values of the mean field components μ_3 and μ_8 at zero temperature. For a degenerate orbit one has to assign $m=0$, or $m=\sqrt{3}\,q$. Evidently the degenerate orbits with $m=0$ are domains of mean field configurations that realize nematic states. Ferromagnetic states are realized on all other orbits of SU(3).

The group SU(4) has four types of orbits: the generic $\frac{\mathrm{SU}(4)}{\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)}$ of dimension 12, the degenerate $\frac{\mathrm{SU}(4)}{\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)}$ of dimension 10, the degenerate $\frac{\mathrm{SU}(4)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2))}$ of dimension 8, and the maximal degenerate $\frac{\mathrm{SU}(4)}{\mathrm{SU}(3)\times\mathrm{U}(1)}$ of dimension 6. Each orbit is defined by numbers m, q, p, which are limiting values of the mean field components μ_3, μ_8, μ_{15}. Almost all orbits are domains of ferromagnetic mean field configurations. Nematic states are realized on the degenerate orbits of dimension 8 as $m=p=0$ and q is arbitrary. So it is probable to reveal a nematic state even in the ferromagnetic region of the phase diagram [6].

As shown above, *limiting values of diagonal components of a mean field* serve as order parameters. Simultaneously, they *define a coadjoint orbit* where the corresponding mean filed configuration lives.

Each orbit possesses a Hamiltonian system with an equation of motion and a group-invariant Hamiltonian. For a degenerate orbit of SU(3) the equation is

$$\frac{\partial \mu_a}{\partial t} = \tfrac{4\mathcal{A}}{3(m^2+q^2)}\, C_{abc}\mu_b(\mu_{c,xx} + \mu_{c,yy}), \qquad (6)$$

where $\mathcal{A}$ denotes a dimensional constant. The values m, q of order parameters (a magnetization and a projection of quadrupole moment) define an orbit via the following equations, which we call constraints:

$$\begin{aligned} &\delta_{ab}\mu_a\mu_b = m^2+q^2 \\ &d_{abc}\mu_b\mu_c = \pm\sqrt{\tfrac{m^2+q^2}{5}}\,\mu_a. \end{aligned}$$

Here $d_{abc} = \frac{\sqrt{3}}{4\sqrt{5}}\,\mathrm{Tr}(\hat{P}_a\hat{P}_b\hat{P}_c + \hat{P}_b\hat{P}_a\hat{P}_c)$ is a symmetric tensor. The corresponding SU(3)-invariant Hamiltonian is the following

$$\mathcal{H}^{\mathrm{deg}}_{\mathrm{eff}} = \tfrac{2\mathcal{J}}{3(m^2+q^2)} \int \sum_{a=1}^{8} \Big((\mu_{a,x})^2 + (\mu_{a,y})^2 \Big)\, dxdy, \tag{7}$$

where the dimensional constant $\mathcal{J}$ has a meaning of exchange integral.

Obviously, equations (6) and (5) coincide. It is easy to show, that (5) coincides with the equation of motion on a maximal degenerate orbit. Recall, that (5) is obtained when correlations between fluctuations of $\{\hat{P}^a_n\}$ are neglected. Presumably, equations of motion on other orbits are derived from (4) via the mean field averaging with more complicate correlation rules.

A generic orbit of SU(3) is determined by the following equations:

$$\begin{aligned} &\delta_{ab}\mu_a\mu_b = m^2+q^2 \\ &d_{abc}\mu_a\mu_b\mu_c = \tfrac{1}{\sqrt{5}}\,q(3m^2-q^2). \end{aligned}$$

The SU(3)-invariant Hamiltonian on this orbit has the form

$$\begin{aligned} \mathcal{H}^{\mathrm{gen}}_{\mathrm{eff}} = \tfrac{\mathcal{J}}{2m^2(m^2-3q^2)^2} \sum_{a=1}^{8} \Big(&(m^2+q^2)^2(\mu_{a,x})^2 + (m^2+q^2)(\eta_{a,x})^2 - \\ &- 2\sqrt{3}\,q(3m^2-q^2)\mu_{a,x}\eta_{a,x} \Big), \end{aligned} \tag{8}$$

where η_a is a quadratic form in $\{\mu_a\}$: $\eta_a = \sqrt{5}\, d_{abc}\mu_b\mu_c$. For more details see [9].

The Hamiltonian systems on coadjoint orbits of SU(3) serve as *classical effective models* for the spin system of $s \geqslant 1$ with biquadratic exchange. Evidently, these models describe large-scale (or slow) fluctuations of the mean field. In this paper we suppose that the order parameters m, q are fixed numbers. But generally speaking, they depend on a temperature T and the interaction constant J. Taking into account these dependencies, one can consider small-scale (or quick) fluctuations of the mean field.

2.4 Geometrical properties of effective Hamitonians

Each coadjoint orbit of a semisimple Lie group is a homogeneous space that admits a Kählerian structure. Thus one can introduce a complex parameterization of an orbit. For this purpose we use a *generalized stereographic projection* (for

more details see [8]). In the case of group SU(3), the projection is represented by the following formulas:

$$\mu_a = -\tfrac{m-\sqrt{3}\,q}{2}\,\zeta_a + m\xi_a, \quad \eta_a = \tfrac{\sqrt{3}\,(m^2-q^2)-2mq}{2}\,\zeta_a + 2mq\xi_a,$$

$$\begin{array}{ll}
\zeta_1 = -\dfrac{w_2+\bar{w}_2+w_3+\bar{w}_3}{\sqrt{2}(1+|w_2|^2+|w_3|^2)} & \xi_1 = -\dfrac{(1-\bar{w}_1)(w_3-w_1w_2)+(1-w_1)(\bar{w}_3-\bar{w}_1\bar{w}_2)}{\sqrt{2}(1+|w_1|^2+|w_3-w_1w_2|^2))} \\
\zeta_2 = i\dfrac{w_3-\bar{w}_3-w_2+\bar{w}_2}{\sqrt{2}(1+|w_2|^2+|w_3|^2)} & \xi_2 = i\dfrac{(1+\bar{w}_1)(w_3-w_1w_2)-(1+w_1)(\bar{w}_3-\bar{w}_1\bar{w}_2)}{\sqrt{2}(1+|w_1|^2+|w_3-w_1w_2|^2)} \\
\zeta_3 = \dfrac{|w_2|^2-|w_3|^2}{1+|w_2|^2+|w_3|^2} & \xi_3 = \dfrac{1-|w_1|^2}{1+|w_1|^2+|w_3-w_1w_2|^2} \\
\zeta_4 = i\dfrac{\bar{w}_2w_3-w_2\bar{w}_3}{1+|w_2|^2+|w_3|^2} & \xi_4 = i\dfrac{w_1-\bar{w}_1}{1+|w_1|^2+|w_3-w_1w_2|^2} \\
\zeta_5 = \dfrac{w_2+\bar{w}_2-w_3-\bar{w}_3}{\sqrt{2}(1+|w_2|^2+|w_3|^2)} & \xi_5 = -\dfrac{(1+\bar{w}_1)(w_3-w_1w_2)+(1+w_1)(\bar{w}_3-\bar{w}_1\bar{w}_2)}{\sqrt{2}(1+|w_1|^2+|w_3-w_1w_2|^2)} \\
\zeta_6 = i\dfrac{w_2-\bar{w}_2+w_3-\bar{w}_3}{\sqrt{2}(1+|w_2|^2+|w_3|^2)} & \xi_6 = i\dfrac{(1-\bar{w}_1)(w_3-w_1w_2)-(1-w_1)(\bar{w}_3-\bar{w}_1\bar{w}_2)}{\sqrt{2}(1+|w_1|^2+|w_3-w_1w_2|^2)} \\
\zeta_7 = -\dfrac{\bar{w}_2w_3+w_2\bar{w}_3}{1+|w_2|^2+|w_3|^2} & \xi_7 = -\dfrac{w_1+\bar{w}_1}{1+|w_1|^2+|w_3-w_1w_2|^2} \\
\zeta_8 = \dfrac{2-|w_2|^2-|w_3|^2}{\sqrt{3}(1+|w_2|^2+|w_3|^2)} & \xi_8 = \dfrac{1+|w_1|^2-2|w_3-w_1w_2|^2}{\sqrt{3}(1+|w_1|^2+|w_3-w_1w_2|^2)}.
\end{array} \tag{9}$$

The coordinates $\{w_1,\, w_2,\, w_3\}$ (Bruhat coordinates according to [7]) parameterize a generic orbit of SU(3). In the case of a degenerate orbit, one has to assign $m=0$ and $w_1=0$, or $m=\sqrt{3}\,q$ and $w_2=0$.

In terms of $\{w_\alpha\}$ the effective Hamiltonians (7) and (8) have the form

$$\mathcal{H}_{\text{eff}} = \mathcal{J}\int\sum_{\alpha,\beta} g_{\alpha\bar{\beta}}\Big(\frac{\partial w_\alpha}{\partial z}\frac{\partial\bar{w}_\beta}{\partial\bar{z}} + \frac{\partial w_\alpha}{\partial\bar{z}}\frac{\partial\bar{w}_\beta}{\partial z}\Big)\,dzd\bar{z}, \quad \alpha,\,\beta = 1,\,2,\,3, \tag{10}$$

where $z = x+iy$ is a complex coordinate on the plane obtained from the atomic lattice after a large-scale limiting process (see Section 2.2). The tensor g is nondegenerate and positively defined, thus it can serve as a metrics on an orbit. Its components $\{g_{\alpha\bar{\beta}}\}$ come from (7) for a degenerate orbit, and from (8) for a generic one. In terms of the auxiliary vector fields $\{\zeta_a\}$ and $\{\xi_a\}$ we have

$$g^{\text{deg1}}_{\alpha\bar{\beta}} = \tfrac{1}{2}\sum_a \frac{\partial\zeta_a}{\partial w_\alpha}\frac{\partial\zeta_a}{\partial\bar{w}_\beta}, \qquad g^{\text{deg2}}_{\alpha\bar{\beta}} = \tfrac{1}{2}\sum_a \frac{\partial\xi_a}{\partial w_\alpha}\frac{\partial\xi_a}{\partial\bar{w}_\beta}\Big|_{w_2=0},$$

$$g^{\text{gen}}_{\alpha\bar{\beta}} = \tfrac{1}{2}\sum_a\left(\frac{\partial\zeta_a}{\partial w_\alpha}\frac{\partial\zeta_a}{\partial\bar{w}_\beta} - \frac{\partial\zeta_a}{\partial w_\alpha}\frac{\partial\xi_a}{\partial\bar{w}_\beta} + \frac{\partial\xi_a}{\partial w_\alpha}\frac{\partial\xi_a}{\partial\bar{w}_\beta}\right).$$

Note, that in terms of $\{w_\alpha\}$ the tensor g does not depend on a particular orbit.

Being a Kählerian manifold an orbit of SU(3) possesses a Kählerian potential. For this purpose we use a potential Φ of the Kirillov-Kostant-Suoriau form:

$$\Phi = m\Phi_1 - \tfrac{m-\sqrt{3}\,q}{2}\,\Phi_2,$$
$$\Phi_1 = \ln(1+|w_1|^2+|w_3{-}w_1w_2|^2), \quad \Phi_2 = \ln(1+|w_2|^2+|w_3|^2),$$

A topological structure of the orbit is characterized by the second cohomology group H^2 of dimension 2. That is why there exist two basis 2-forms, for example generated by the potentials Φ_1, Φ_2. Each of them defines a topological charge

$$\mathcal{Q}_k = \frac{1}{4\pi}\int\sum_{\alpha,\beta}\frac{i\partial^2\Phi_k}{\partial w_\alpha\partial\bar{w}_\beta}\Big(\frac{\partial w_\alpha}{\partial z}\frac{\partial\bar{w}_\beta}{\partial\bar{z}} - \frac{\partial w_\alpha}{\partial\bar{z}}\frac{\partial\bar{w}_\beta}{\partial z}\Big)\,dz\wedge d\bar{z}, \quad k=1,\,2.$$

On a degenerate orbit only one potential is governing, and only one topological charge exists. Then the expressions for $\mathcal{Q}_k$ and $\mathcal{H}_{\rm eff}^{\deg k}$ differ only in a sign. Evidently,

$$\mathcal{H}_{\rm eff}^{\deg k} \geqslant 4\pi\mathcal{J}|\mathcal{Q}_k|. \tag{11}$$

Hence, *on a degenerate orbit a minimum of $\mathcal{H}_{eff}$ is realized if the equality holds, that takes place if $\{w_\alpha\}$ are holomorphic or antiholomorphic functions.* Here we use an idea of Belavin and Polyakov [2].

For a generic orbit we define a topological charge by $\mathcal{Q}=\mathcal{Q}_1+\mathcal{Q}_2$. In order to extend inequality (11) to generic orbits with the topological charge $\mathcal{Q}$, we construct an effective Hamiltonian by the following formula:

$$\mathcal{H}_{\rm eff}^{\rm gen} = \tfrac{\mathcal{J}}{2m^2(m^2-3q^2)^2}\sum_{a=1}^{8}\Big(C_1(\mu_{a,x})^2 + C_2(\eta_{a,x})^2 + C_3\mu_{a,x}\eta_{a,x}\Big), \tag{12}$$

$$\begin{aligned} C_1 &= m^4+q^4-\tfrac{4}{\sqrt{3}}\,mq(q^2-m^2)+14m^2q^2,\\ C_2 &= \tfrac{5}{3}\,m^2+q^2-\tfrac{2}{\sqrt{3}}\,mq,\\ C_3 &= \tfrac{2}{\sqrt{3}}\,m^3+2q^3-\tfrac{26}{3}\,m^2q+\tfrac{2}{\sqrt{3}}\,mq^2.\end{aligned}$$

In terms of $\{w_\alpha\}$ it is reduced to the form (10) with the metrics

$$g_{\alpha\bar{\beta}}^{\rm gen} = \tfrac{1}{2}\sum_a\left(\frac{\partial\zeta_a}{\partial w_\alpha}\frac{\partial\zeta_a}{\partial\bar{w}_\beta} + \frac{\partial\xi_a}{\partial w_\alpha}\frac{\partial\xi_a}{\partial\bar{w}_\beta}\right).$$

Then we get

$$\mathcal{H}_{\rm eff}^{\rm gen} \geqslant 4\pi\mathcal{J}|\mathcal{Q}|,$$

and *a minimum of $\mathcal{H}_{eff}^{gen}$ is realized if the equality holds, that takes place if $\{w_\alpha\}$ are holomorphic or antiholomorphic functions.*

3 Large-scale topological excitations

Now we construct a particular class of topological excitations that give minimums to the effective Hamiltonians $\mathcal{H}_{\rm eff}^{\deg 1}$, and $\mathcal{H}_{\rm eff}^{\rm gen}$ defined by (12). We describe these excitations by holomorphic functions $\{w_\alpha(z)\}$. Each set $\{w_1(z)$, $w_2(z)$, $w_3(z)\}$ represents a mean field configuration of the system in question.

First, we consider a degenerate orbit of SU(3) with $m=0$, $q=-\frac{2}{\sqrt{3}}$, where a nematic state is realized. In order to satisfy the limiting conditions: $\mu_3\to m$, $\mu_8\to q$, the functions $\{w_\alpha(z)\}$ have to vanish as $z\to\infty$. Let

$$w_1(z)=0, \quad w_2(z)=\tfrac{a_2}{z-z_2}, \quad w_3(z)=\tfrac{a_3}{z-z_3}, \qquad a_2,\,z_2,\,a_3,\,z_3\in\mathbb{C}, \tag{13}$$

be a large-scale excitation in the magnet in question. The corresponding mean field configuration is obtained by substitution of (13) into (9). A behavior of the components μ_3 and μ_8 is represented on the Fig. 1. As $z \to \infty$ the values of μ_3, μ_8 tend to m, q.

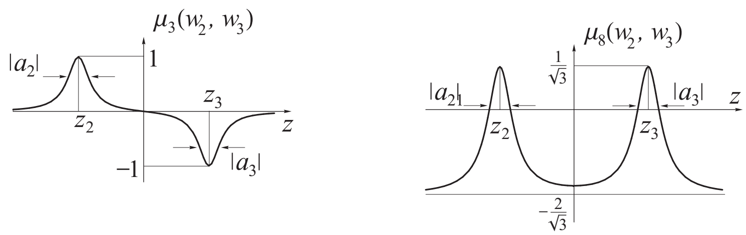

Fig. 1. Profiles of the mean field components μ_3, μ_8 in the case of configuration (13).

Suppose $z_2 = z_3$. By shifting of the coordinate z one can easily reduce (13) to the configuration

$$w_1(z) = 0, \quad w_2(z)=\tfrac{a_2}{z}, \quad w_3(z)=\tfrac{a_3}{z}, \qquad a_2,\, a_3 \in \mathbb{C},$$

and calculate its topological charge:

$$\mathcal{Q} = \frac{2i}{4\pi} \iint_{\mathbb{C}} \frac{(a_2^2 + a_3^2)}{(|z|^2 + a_2^2 + a_3^2)^2}\, dz \wedge d\bar{z} = 1. \tag{14}$$

In the case of $z_2 \neq z_3$ the topological charge equals 2.

It can be interpreted as follows. Each pole of a mean field configuration represents a kind of Belavin-Plyakov soliton. Each soliton gives a unit topological charge. Thus, two distinct solitons have the topological charge 2. No continuous deformation take a configuration of topological charge 2 to a configuration of topological charge 1. If we allow noncontinuous deformation, then two solitons can meet at any point and join into one, at the same time an energy is released. From (11), it follows that the released energy equals $4\pi\mathcal{J}$ per one pole.

Note, that the energy of configuration (13) does not depend on parameters of solitons: a_2, z_2, a_3, z_3. It means that the excitation can grow (when $|a_2|$ and $|a_3|$ grow) preserving an energy. Such growth immediately leads to destruction of an order in the system.

One can construct a configuration with more than two solitons:

$$w_1(z) = 0, \quad w_2(z) = a_2 / \prod_{k=1}^{n} (z - z_{2k}), \quad w_3(z) = a_3 / \prod_{k=1}^{m} (z - z_{3k}), \tag{15}$$

here a_2, a_3, $\{z_{2k}\}_{k=1}^{n}$, and $\{z_{3k}\}_{k=1}^{m}$ are fixed complex numbers. If all values $\{z_{2k}\}_{k=1}^{n}$ and $\{z_{3k}\}_{k=1}^{m}$ are distinct, a topological charge equals $n+m$. When a pole of the function $w_2(z)$ coincides with a pole of $w_3(z)$, the topological charge decreases by 1. But a coincidence of two poles of the same function (for example

$w_2(z)$) does not lead to a decrease of the topological charge. It is easy to see, that the minimal energy of configuration (15) equals $4\pi\mathcal{J} \cdot \min(n, m)$.

Now we consider a generic orbit, where a ferromagnetic state is realized. Suppose $m=1$ and $q=-\frac{2}{\sqrt{3}}$. In this case, we describe a mean field in the magnet by the effective Hamiltonian $\mathcal{H}_{\text{eff}}^{\text{gen}}$, defined by (12). Let

$$w_1(z) = \tfrac{a_1}{z-z_1}, \quad w_2(z)=\tfrac{a_2}{z-z_2}, \quad w_3(z)=\tfrac{a_3}{z-z_3}, \quad a_k, z_k \in \mathbb{C}, \quad k = 1, 2, 3. \qquad (16)$$

be a large-scale excitation of the mean field. A calculation of topological charges gives: 1) $\mathcal{Q}_1=3$, $\mathcal{Q}_2=2$, if $a_1 \neq a_2 \neq a_3$ or $a_1=a_2 \neq a_3$, 2) $\mathcal{Q}_1=2$, $\mathcal{Q}_2=2$, if $a_1=a_3 \neq a_2$, 3) $\mathcal{Q}_1=2$, $\mathcal{Q}_2=1$, if $a_1=a_2=a_3$ or $a_1 \neq a_2=a_3$.

4 Conclusion and discussion

Each generalized Heisenberg Hamiltonian with high powers of the exchange interaction can be reduced to a bilinear form. By a mean field averaging we obtain a classical system from the original quantum one. An averaging of the Heisenberg equation gives a Landau-Lifshitz like equation for a mean field. Using Lie group apparatus, we construct effective Hamiltonians for the classical system with SU(3) symmetry. One of them $\mathcal{H}_{\text{eff}}^{\text{deg}}$ is an SU(3)-analogue of the Hamiltonian commonly used in theory of magnetism. In addition, we propose another one $\mathcal{H}_{\text{eff}}^{\text{gen}}$, which is biquadratic in the mean field. Further, we construct examples of topological excitations that give minimums to the Hamiltonians. Such excitations can change their shapes and grow preserving an energy. This is a probable scenario for the destruction of an ordered state in a 2D magnet at nonzero temperature, that agrees with the Mermin-Wagner theorem.

5 Acknowledgments

This work is partly supported by the grant of the International Charitable Fund for Renaissance of Kyiv-Mohyla Academy.

References

[1] Mermin N. D., Wagner H. *Phys. Rev. Lett* **17** 1133 (1966).

[2] Belavin A. A., Polyakov A. M. *JETP Letters* **22** 245 (1975).

[3] Bogolyubov N N and Bogolyubov N N (Jr.) *Introduction to Quantum Statistical Mechanics [in Russian]* (Moscow: Nauka) 1984 p 384

[4] Matveev V M *Zh. Eksp. Teor. Fiz.* **65** 1626 (1973)

[5] Nauciel-Bloch M, Sarma G and Castets A *Phys. Rev. B* **5** 4603 (1972)

[6] Buchta K., Fath G., Legeza Ö., Solyom J. *Phys. Rev. B* **72** 054433 (2005)

[7] Picken R. F. *J. Math. Phys.* **31** 616 (1990).

[8] Bernatska J., Holod P. *Proceedings of the IXth International Conference on Geometry, Integrability and Quantization* (Sofia) 2008, p. 146–166; arXiv: math.RT/0801.2913

[9] Bernatska J., Holod P. *J. Phys. A* **42** (to be published in 2009)